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A Semi-Implicit Numerical Model for Baroclinic Oceans

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A multilevel, free-surface, viscous primitive equation ocean model has been developed to study the response of baroclinic oceans to stationary and moving weather systems. The model avoids the usual small time steps associated with the fast moving surface gravity waves by dividing the flow into baroclinic and barotropic (vertically averaged) modes, with the surface waves coupled to the latter; the baroclinic modes are then treated explicitly and the barotropic waves implicitly. Results of a 60-hr time integration for a stationary hurricane are presented, and compared to the results of an integration performed with purely explicit time marching techniques.

1. INTRODUCTION

The equations representing a baroclinic ocean exclude sound waves through the incompressibility approximation, but retain fast moving surface gravity waves with speeds of ∼500 km/hr or so, unless the rigid lid approximation is involved. Recent studies have indicated that certain aspects of baroclinic wave growth and geostrophic adjustment processes are misrepresented by the rigid lid approximation [3]. On the other hand, the fast free surface waves have limited the time steps employed by up-to-date ocean models to ∼15 min with a 100 km resolution. It was to keep the free surface aspect of ocean modelling without its severe limitation of the time marching step that the present split-mode, semi-implicit model was developed.

Kwizak and Robert [4] have developed a semi-implicit technique for atmospheric motions by treating the gravity waves implicitly, all other modes explicitly. O'Brien and Hurlburt [7] designed a two-layer semi-implicit ocean model to study the coastal upwelling. However, in both of these models, a Helmholtz equation for one of the appropriate variables had to be solved in each layer, offsetting some of the computer time savings achieved by the larger time step. In Section 2 we present the governing

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equations and in Section 3 their splitting into barotropic and baroclinic modes. The model presented in this paper solves only one Helmholtz equation for the surface height, which is really a great advantage in this case, because at least seven vertical mesh points (or levels) are needed to resolve the thermocline and its changes in typical situations of interest. The fixed mesh point or level model can be looked upon as an Eulerian approach to treating vertical transport processes, whereas layer models can be regarded as a Lagrangian approach, with the layer thickness following thermocline deformations. In the case of the atmosphere, there are usually $n - 1$ coupled Helmholtz equations for an n-level model, and, in the ocean, n coupled equations for an n-layer model.

In Section 4 the numerical techniques for the spatial differencing and time integration will be presented. In Section 5 the initial and boundary conditions for the particular hurricane simulation and comparison study are given. In Section 6 the results of a 60-hour time integration for the case of a stationary hurricane with a 15-level model are given, and compared to similar integrations with a purely explicit time marching technique.

2. Physical Model and Basic Equations

The ocean we want to model is assumed to be hydrostatic, Boussinesq, and contained in a rectangular basin with a flat bottom. The relevant equations of momentum and heat transport can then be written as (for definition of symbols, see Appendix A)

$$\frac{Du}{Dt} = fv - \frac{1}{\rho_0}\frac{\partial P}{\partial x} + \frac{\partial}{\partial x} K_{\mathrm{H}} \frac{\partial u}{\partial x} + \frac{\partial}{\partial y} K_{\mathrm{H}} \frac{\partial u}{\partial y} + \frac{\partial}{\partial z} K_{\mathrm{V}} \frac{\partial u}{\partial z}, \tag{1}$$

$$\frac{Dv}{Dt} = -fu - \frac{1}{\rho_0}\frac{\partial P}{\partial y} + \frac{\partial}{\partial x} K_{\mathrm{H}} \frac{\partial v}{\partial x} + \frac{\partial}{\partial y} K_{\mathrm{H}} \frac{\partial v}{\partial y} + \frac{\partial}{\partial z} K_{\mathrm{V}} \frac{\partial v}{\partial z}, \tag{2}$$

$$\frac{\partial P}{\partial z} = -\rho g, \tag{3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{4}$$

$$\frac{DT}{Dt} = \frac{\partial}{\partial x} A_{\mathrm{H}} \frac{\partial T}{\partial x} + \frac{\partial}{\partial y} A_{\mathrm{H}} \frac{\partial T}{\partial y} + \frac{\partial}{\partial Z} A_{\mathrm{V}} \frac{\partial T}{\partial Z}, \tag{5}$$

$$\frac{DS}{Dt} = \frac{\partial}{\partial x} B_{\mathrm{H}} \frac{\partial s}{\partial x} + \frac{\partial}{\partial y} B_{\mathrm{H}} \frac{\partial s}{\partial y} + \frac{\partial}{\partial z} B_{\mathrm{V}} \frac{\partial s}{\partial z}, \tag{6}$$

$$\frac{\partial h}{\partial t} = -H\left\{\frac{\partial [u]}{\partial x} + \frac{\partial [v]}{\partial y}\right\} - \left\{\frac{\partial}{\partial x}(u_1 h) + \frac{\partial}{\partial y}(v_1 h)\right\}. \tag{7}$$

We use the equation of state [2]

$$\rho = \frac{P_1 + P_0}{1.000027(\lambda + \alpha_0(P' + P_0))}, \tag{8a}$$

where

$$\alpha_0 = 0.698,$$

$$\lambda = 1779.5 + 11.25T_1 - 0.0745(T_1)^2 - (3.8 + 0.01T_1)S, \quad \text{(b8)}$$

$$P_1 = 5890 + 38T_1 - 0.375T_1^2 + 3S, \quad \text{(8c)}$$

with P_0 being the total pressure and T_1 the temperature deviation in units of atmospheres and ° C, respectively.

K_H, A_H, and B_H are the horizontal eddy diffusivities for momentum, temperature and salinity, respectively, and K_V, A_V, and B_V the corresponding vertical components.

3. Mode Splitting for Semi-implicit Formulation

We can rewrite Eqs. (1) and (2) as

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial h}{\partial x} - \frac{1}{\rho_0}\frac{\partial P_a}{\partial x} + A, \quad (9)$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial h}{\partial y} - \frac{1}{\rho_0}\frac{\partial P_a}{\partial y} + B, \quad (10)$$

where

$$A = -\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)u - \frac{1}{\rho_0}\frac{\partial P'}{\partial x} + \left(\frac{\partial}{\partial x}K_H\frac{\partial}{\partial x} + \frac{\partial}{\partial y}K_H\frac{\partial}{\partial y} + \frac{\partial}{\partial z}K_V\frac{\partial}{\partial z}\right)u, \quad \text{(11a)}$$

$$B = -\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)v - \frac{1}{\rho_0}\frac{\partial P'}{\partial y} + \left(\frac{\partial}{\partial x}K_H\frac{\partial}{\partial x} + \frac{\partial}{\partial y}K_H\frac{\partial}{\partial y} + \frac{\partial}{\partial z}K_V\frac{\partial}{\partial z}\right)v, \quad \text{(11b)}$$

with

$$P' = P - \rho_0 gh - P_a - \rho_0 gz. \quad \text{(11c)}$$

Here P' is the pressure due to the baroclinicity of the ocean (i.e., its density perturbation), P_a is atmospheric pressure, and h is surface height, above the mean height H, P the total pressure in the water.

Averaging Eqs. (9) and (10) with respect to height $(1/H)\int_0^H (\ )\,dz = [\ ]$, and assuming that h, the surface deviation, is at least two orders of magnitude smaller than H, the total (mean) depth of the ocean, we have

$$\frac{\partial[u]}{\partial t} - f[v] = -g\frac{\partial h}{\partial x} - \frac{1}{\rho_0}\frac{\partial P_a}{\partial x} + [A], \quad (12)$$

$$\frac{\partial[v]}{\partial t} + f[u] = -g\frac{\partial h}{\partial y} - \frac{1}{\rho_0}\frac{\partial P_a}{\partial y} + [B]. \quad (13)$$

Subtracting (12) and (13) from (9) and (10) we have

$$\frac{\partial u'}{\partial t} - fv' = A - [A], \tag{14}$$

$$\frac{\partial v'}{\partial t} + fu' = B - [B]. \tag{15}$$

We note here that Eqs. (14) and (15) are independent of the terms that govern external gravity waves, namely, $-g\nabla h$ and $-(1/\rho_0)\, \nabla P_a$. Thus, Eqs. (14), (15), together with (3), (4), (5), and (6) govern the slow moving baroclinic modes, mostly Rossby waves and internal waves. Equations (12), (13), and (7) govern the external gravity modes and can be solved implicitly.

4. Numerical Model

In this section we will present the finite difference techniques employed to solve the two systems of equations given in Section 3. We shall represent values of a dependent variable $\phi(x, y, z, t)$ as discrete values of the independent variables $x = i\Delta x$, $y = j\Delta y$, $z = k\Delta z$, and $t = n\Delta t$ as ϕ^n_{ijk}. The finite difference operators that replace first and second derivatives are the following.

$$\delta_x\phi = (\phi_{i+1/2} - \phi_{i-1/2})/\Delta x, \tag{16a}$$

$$\delta_{2x}\phi = (\phi_{i+1} - \phi_{i-1})/2\Delta x, \tag{16b}$$

$$\delta_x{}^2\phi = \delta_x(\delta_x\phi) = (\phi_{i+1} + \phi_{i-1} - 2\phi_i)/\Delta x^2, \tag{16c}$$

$$\bar{\phi}^x = (\phi_{i+1/2} + \phi_{i-1/2})/2. \tag{16d}$$

The last expression (16d) is used to define values of ϕ at spatial locations halfway between the original grids; these are needed in the differencing of the transport terms for the various quantities (see Figs. 1 and 2).

We shall first treat the system governing the barotropic motion, Eqs. (12), (13), and (7).

$$[u]^{n+1} = [u]^{n-1} + 2\Delta t\,\{(F^{n+1} + F^{n-1})/2 + [A]^n\}, \tag{17}$$

$$[v]^{n+1} = [v]^{n-1} + 2\Delta t\,\{(G^{n+1} + G^{n-1})/2 + [B]^n\}, \tag{18}$$

$$h^{n+1} = h^{n-1} + 2\Delta t\,\{(J^{n+1} + J^{n-1})/2 + 2\Delta t\, K^n\}, \tag{19}$$

where

$$F = f[v] - g\delta_x\bar{h}^y - (1/\rho_0)\, \delta_x\bar{P}_a{}^x, \tag{20a}$$

$$G = -f[u] - g\delta_y\bar{h}^x - (1/\rho_0)\, \delta_y\bar{P}_a{}^y, \tag{20b}$$

$$J = -H\{\delta_x[\bar{u}]^y + \delta_y[\bar{v}]^x\}, \tag{20c}$$

$$K = -\{\delta_x(\bar{u}^y\bar{h}^x) + \delta_y(\bar{v}^x\bar{h}^y)\}. \tag{20d}$$

All terms in a given equation are expanded about the spatial location on which the

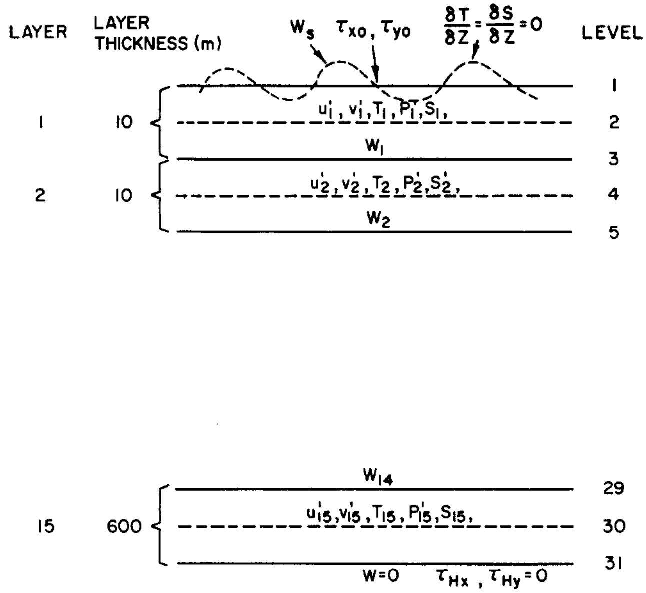

FIG. 1. Vertical layering of the model.

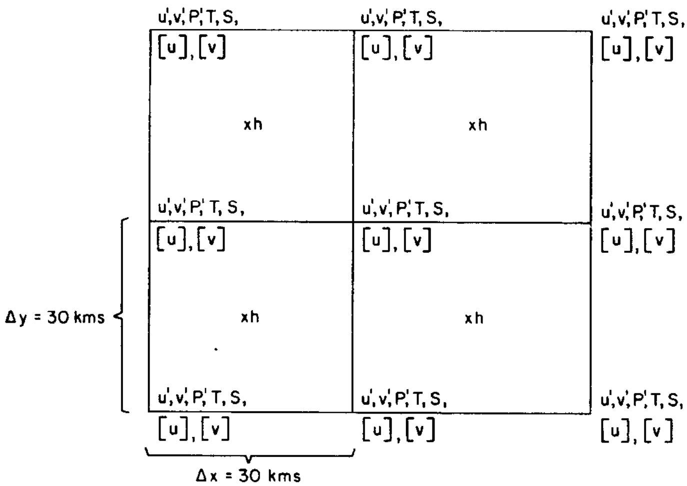

FIG. 2. Grid network on a horizontal plane.

respective variable occuring on the left-hand side is defined. Note that u, v and u', v' are all defined at the same lattice points, so that no averaging is necessary for the Coriolis terms.

We now assume that the pressure P_a of the atmosphere at sea level is known.

Because of the simple way that $[u]$ and $[v]$ enter F and G, we can then find direct expressions for $[u]^{n+1}$ and $[v]^{n+1}$.

$$[u]^{n+1} = \frac{1}{(1+f^2\Delta t^2)}\{(-g\Delta t)(\delta_x\bar{h}^y + f\Delta t\delta_y\bar{h}^x)^{n+1} + \gamma_1{}^n + f\Delta t\gamma_2{}^n\}, \quad (21a)$$

$$[v]^{n+1} = \frac{1}{(1+f^2\Delta t^2)}\{(-g\Delta t)(\delta_y\bar{h}^x - f\Delta t\delta_x\bar{h}^y)^{n+1} + \gamma_2{}^n - f\Delta t\gamma_1{}^n\}, \quad (21b)$$

with

$$\gamma_1{}^n = 2\Delta t\{[A]^n - (1/\rho_0)\,\overline{\delta_x\bar{P}_a{}^x}^{2t}\} + [u]^n + f\Delta t[v]^n - g\Delta t(\delta_x\bar{h}^y)^n, \quad (22a)$$

$$\gamma_2{}^n = 2\Delta t\{[B]^n - (1/\rho_0)\,\overline{\delta_y\bar{P}_a{}^y}^{2t}\} + [v]^n - f\Delta t[u]^n - g\Delta t(\delta_y\bar{h}^x)^n. \quad (22b)$$

Substituting expressions (21a, b) into (19) we get a second-order elliptic equation for h^{n+1};

$$h^{n+1} - Hg\Delta t^2\left\{\left(\frac{1}{1+f^2\Delta t^2}\,\overline{\delta_x(\delta_x\bar{h}^y + f\Delta t\delta_y\bar{h}^x)}^y\right)^{n+1} + \left(\delta_y\frac{1}{1+f^2\Delta t^2}\,\overline{(\delta_y\bar{h}^x - f\Delta t\delta_x\bar{h}^y)}^x\right)^{n+1}\right\}$$
$$= E^n - H\Delta t\left\{\frac{1}{1+f^2\Delta t^2}\,\overline{\delta_x(\gamma_1{}^n + f\Delta t\gamma_2{}^n)}^y + \delta_y\frac{1}{1+f^2\Delta t^2}\,\overline{(\gamma_2{}^n - f\Delta t\gamma_1{}^n)}^x\right\}, \quad (23)$$

where

$$E^n = \Delta t\,J^{n-1} + 2\Delta t\,K^n + h^{n-1}, \quad (24)$$

and $\gamma_1{}^n$, $\gamma_2{}^n$, J, and K are defined by 22a, 22b, 20c, and 20d, respectively.

Equation (23) is solved iteratively, subject to the boundary conditions that $h \to 0$ at the far boundaries.

Then (21) can be used to find u^{n+1} and v^{n+1}, by substituting h^{n+1} from (23).

The finite differencing of the system of equations (14) and (15), governing the motion of the baroclinic modes, follows the procedure used for the barotropic system. We rewrite them as

$$\partial u'/\partial t = fv' + A - [A] = M + F_x, \quad (25)$$

$$\partial v'/\partial t = fu' + B - [B] = N + F_y, \quad (26)$$

where

$$M = -(\mathbf{v}\cdot\nabla)u - (1/\rho_0)(\partial P'/\partial x) - [A] + fv', \quad (27)$$

$$N = -(\mathbf{v}\cdot\nabla)v - (1/\rho_0)(\partial P'/\partial y) - [B] - fu', \quad (28)$$

with F_x, F_y being the friction terms defined in (11). The time differencing of (25) and (26) becomes an explicit, leap-frog technique,[1] i.e.,

$$(u')^{n+1} = (u')^{n-1} + 2\Delta t(M^n + F_x^{n-1}), \tag{29}$$

$$(v')^{n+1} = (v')^{n-1} + 2\Delta t(N^n + F_y^{n-1}). \tag{30}$$

The spacial differencing of the terms M and N is given in Appendix B. It should be noted that the finite difference form of the advective terms has quadratic conservative property and therefore the model is free from nonlinear instability. The finite difference form of the continuity and hydrostatic equations are given by,

$$\delta_x u + \delta_y v + \delta_z w = 0, \tag{31a}$$

and

$$\delta_z P = -g\bar{\rho}^z. \tag{31b}$$

5. Initial and Boundary Conditions

Initially, the ocean is assumed to be barotropic with no motion. The vertical distribution of the temperature and salinity for the undisturbed model ocean is given in Table I. These represent mean values of the North Atlantic Ocean for the month of August [8].

TABLE I

Mean values of Temperature, Salinity, and Density for the Month of August at Various Depths of the North Atlantic Ocean

Level	Depth (m)	Temperature (°K)	Salinity (g kg^{-1})	Density (g cm^{-3})
2	5	298.3	36.1	1.0243
4	15	297.7	36.1	1.0245
6	25	296.8	36.2	1.0249
8	35	295.9	36.3	1.0253
10	45	295.0	36.4	1.0256
12	60	293.8	36.4	1.0260
14	80	292.5	36.4	1.0264
16	100	291.6	36.5	1.0268
18	120	290.9	36.5	1.0270
20	140	290.4	36.5	1.0272
22	175	289.7	36.5	1.0276
24	250	289.3	36.5	1.0280
26	575	287.8	36.1	1.0295
28	1125	286.3	35.5	1.0318
30	1700	285.5	34.9	1.0341

[1] The leap-frog time-splitting was eliminated by averaging the solutions after every 30 time steps.

A steady state, stationary axisymmetric hurricane in gradient wind balance is assumed to be the driving force at the ocean surface. The pressure distribution of the model hurricane is given by

$$P_{\mathrm{A}}(r) = P_0 + (P_n - P_0)\, e^{-R/r}, \tag{32}$$

where R is the radius of the maximum wind, P_0 is the central pressure of the hurricane and P_n is the surface pressure of the mean tropical atmosphere. This equation was used by O'Brien and Reid [6] to study the response of a two layer ocean to a steady state hurricane. In this model we use

$$P_n = 1015 \text{ mbar},$$
$$P_0 = 960 \text{ mbar},$$

and

$$R = 30 \text{ kM}.$$

The magnitude of the tangential velocity (v_0) that balances the pressure distribution given by (32) is obtained by the solving the gradient wind equation, given by

$$\frac{V_0^2}{r} + fV_0 = \frac{1}{\rho_{\mathrm{a}}} \frac{\partial P}{\partial r}, \tag{33}$$

where ρ_{a} is the density of the mean tropical atmosphere at the surface. The radial velocities are assumed to be inward everywhere with magnitudes 0.3 times the magnitude of the tangential velocity at the same point. The stress at the ocean surface due to the overlying hurricane is obtained by using the following simple formulas.

$$\tau_{\theta 0} = \rho_{\mathrm{a}} C_{\mathrm{D}} V_\theta (V_\theta^2 + V_{\mathrm{r}}^2)^{1/2}, \qquad -(15)$$

$$\tau_{\mathrm{r}0} = \rho_{\mathrm{a}} C_{\mathrm{D}} V_{\mathrm{r}} (V_\theta^2 + V_{\mathrm{r}}^2)^{1/2}, \qquad -(16)$$

and

$$\tau_0 = (\tau_{\theta 0}^2 + \tau_{\mathrm{r}0}^2)^{1/2},$$

where τ_0 is the total stress, $\tau_{\theta 0}$ and $\tau_{\mathrm{r}0}$ are, respectively, the tangential and radial components of τ_0, and C_{D} is the drag coefficient. C_{D} is assumed to have a constant value of 3×10^{-3}. The values of P, τ_0, $\tau_{\theta 0}$, and $\tau_{\mathrm{r}0}$ are plotted as a function of radius in Figs. 3 and 4.

The vertical component of the stresses τ_{xz}, τ_{yz} are assumed to be of the following form.

$$\tau_{xz} = \rho_0 K_z \frac{\partial u}{\partial z},$$

$$\tau_{yz} = \rho_0 K_z \frac{\partial v}{\partial z},$$

in the interior of the ocean, and at the surface they are obtained from Eqs. (15) and

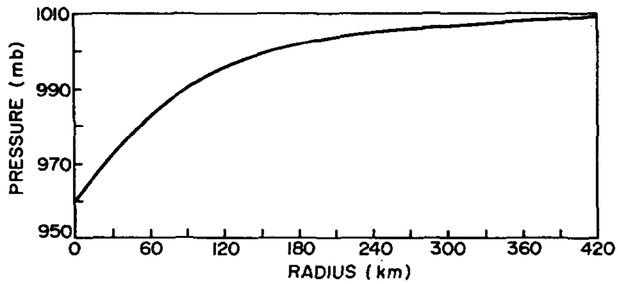

FIG. 3. Hurricane pressure distribution as a function of radius.

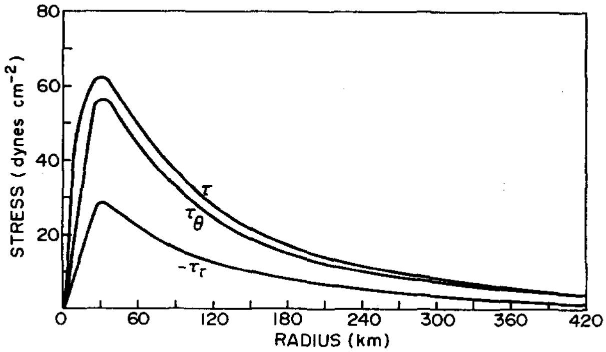

FIG. 4. Hurricane stress distribution as a function of radius.

(16) where u_A and v_A are the E–W and N–S components of the atmospheric (hurricane) velocity at the ocean surface, ρ_a is the density of the mean tropical atmosphere at the surface of the ocean and C_D is the drag coefficient, for which a constant of value of 3×10^{-3} has been assumed.

The vertical eddy coefficients of momentum, K_Z, and heat, A_Z, will depend upon the Richardson's number (R_i). In the present model, these values are computed from the formulas given by Munk and Anderson [5].

$$K_z = K_0(1 + \beta_a R_i)^{-K_V}$$

and

$$A_z = K_0(1 + \beta_T R_i)^{-K_T},$$

where K_0 is the eddy coefficient of momentum for a homogeneous ocean and R_i is the Richardson's number. As recommended by Munk [5], the following values are used for the constants K_0, β_V, β_T, K_V, and K_T.

$$K_0 = 10^{-3}, \quad \beta_V = 10, \quad \beta_T = 10/3, \quad K_V = 0.5, \quad \text{and} \quad K_T = 1.5.$$

The stresses at the bottom of the ocean are assumed to vanish at all times.

6. Results and Comparison for Three-dimensional Ocean Simulation

We have performed two numerical experiments on the same physical model, with the initial and boundary conditions, given in Section 5, utilizing an explicit and implicit model, respectively. The formulation of the explicit model is very straightforward; e.g., see [1].

Figures 5 and 6 illustrate the comparison of the implicit and explicit time integrations

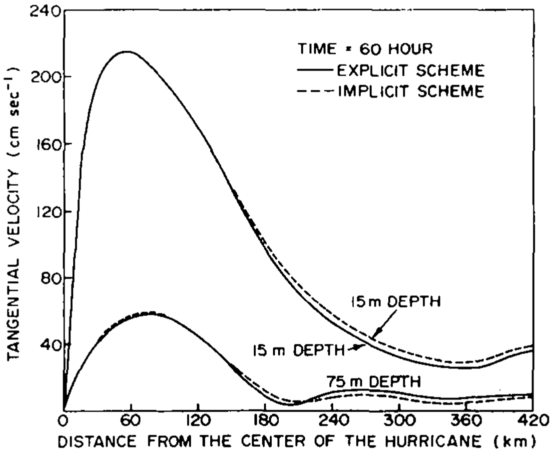

FIG. 5. A comparison of the tangential velocities (15 and 75 m depths) obtained at the end of 60 hours of integration of the implicit and explicit models.

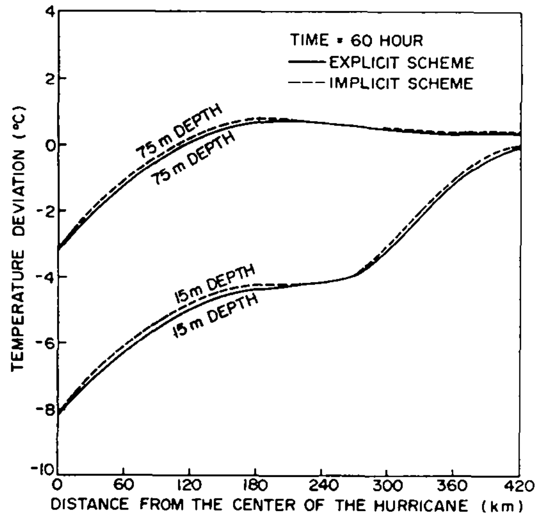

FIG. 6. A comparison of the deviatory temperatures (15 and 75 m depths) obtained at the end of 60 hours of integration of the implicit and explicit models.

for the azimuthal velocity and temperature deviations at various depths, respectively. The results are illustrated at 60 hours of elapsed integration as a function of radial distance. The velocity results show appreciable deviation only near the boundaries, particularly in the lower layer. The errors for h in the upper layer amount to less than 10 %, but amount to about 25 % in the lower layer. The temperature results show a smaller error, a few percent, in all regions of the flow. We attribute the velocity discrepancies to the different arrival times and reflection of the surface waves as treated by the implicit and explicit model, respectively. We plan to insert sponge layers containing a Rayleigh viscosity to reduce reflection effects. The errors seem to have the same absolute size in each layer, and can be attributed to surface effects manifested in layers 3 and 7. The amount of computer time saving achieved is a factor of 12; the time-step difference of factor 15 was slightly offset by the time necessary to iterate the one two-dimensional Helmholtz equation for the free surface height.

APPENDIX A: List of Symbols

A_H, A_V	Horizontal and vertical eddy diffusivities
B_H, B_V	Horizontal and vertical eddy diffusivities for salinity
C_D	Drag coefficient
f	Coriolis parameter
g	Acceleration due to gravity
h	Height of the free surface (Positive for crests and negative for troughs)
H	Total depth of the undisturbed ocean
K_H, K_V	Horizontal and vertical eddy diffusivities for momentum
P	Total pressure
P_0	Total pressure in atmospheres
P'	Pressure due to baroclinicity of the ocean
P_a	Atmospheric pressure at the surface of the ocean
R_i	Richardson's number
S	Salinity (parts per thousand)
T	Temperature (° K)
T_1	Temperature (° C)
$[T]$	Vertically averaged temperature
T'	Deviatory temperature ($T - [T]$)
u, v, w	E–W, N–S, and vertical components of the velocity
$[u]$, $[v]$	Vertically averaged values of u and v
u', v'	Velocity deviations ($u - [u]$, $v - [v]$)
u_1, v_1	Magnitudes of u and v velocities in the topmost layer
x, y	Coordinates in the E–W and N–S directions (positive E, N)

z	Vertical coordinate (positive downward)
Δx, Δy	Grid distances in x and y directions
Δz	Layer thickness
ρ	Density of the ocean
ρ_0	Mean density of the ocean
ρ_s	Density of the ocean at the surface
ρ_a	Atmospheric density at the air–sea interface
[ ]	$= (1/H) \int_0^H (\ ) dz$ Vertical average

APPENDIX B: Finite Difference Form of the Advective Terms in the Governing Equations (1), (2), (5), and (6)

The finite difference form of the advective terms in the governing equations for momentum, temperature and salinity are given by

$$\delta_x(\bar{u}^x\bar{u}^x) + \delta_y(\bar{u}^y\bar{v}^y) + \delta_z(w\bar{u}^z), \tag{B.1}$$

$$\delta_x(\bar{u}^x\bar{v}^x) + \delta_y(\bar{v}^y\bar{v}^y) + \delta_z(w\bar{v}^z), \tag{B.2}$$

$$\delta_x(\bar{u}^x\bar{T}^x) + \delta_y(\bar{v}^y\bar{T}^y) + \delta_z(w\bar{T}^z), \tag{B.3}$$

and

$$\delta_x(\bar{u}^x\bar{s}^x) + \delta_y(\bar{v}^y\bar{s}^y) + \delta_z(w\bar{s}^z). \tag{B.4}$$

Acknowledgment

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Reference

1. W. P. Crowley, *J. Computational Phys.* **3** (1968), 111–147.
2. C. Eckart, *Amer. J. Sci.* **256** (1958), 225–240.
3. W. L. Gates, *J. Atmos. Sci.* **25** (1968), 3–22.
4. M. Kwizak and J. A. Robert, *Mon. Wea. Rev.* **99** (1971), 32–36.
5. W. H. Munk and E. R. Anderson, *J. Marine Res.* **7** (1948), 276–295.
6. J. J. O'Brien and R. O. Reid, *J. Atmos. Sci.* **24** (1967), 197–207.
7. J. J. O'Brien and H. E. Hurlburt, *J. Phys. Ocean.* **2** (1972), 14–26.
8. U. S. Navy Oceanographic Atlas of the North Atlantic Ocean. Publication No. 700., U. S. Naval Oceanographic Office, Washington, D.C., 1967.